

# Boundary classes of graphs for the dominating set problem

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Received 10 March 2003; accepted 13 April 2004

## Abstract

The notion of a boundary class has been recently introduced as a tool for classification of hereditary classes of graphs according to the time complexity of NP-hard graph problems. In the present paper we concentrate on the dominating set problem and obtain three boundary classes for it.

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**Keywords:** Dominating set; Hereditary class; Computational complexity

## 1. Introduction

Typically, a problem, which is NP-hard in general graphs, becomes tractable (solvable in polynomial time) when restricted to some particular classes of graphs. A helpful tool for classification of graph classes according to the time complexity of a given NP-hard problem is the notion of a boundary class. Originally it has been introduced with respect to the independent set problem [2]. In the present paper, we define this notion in its general form in Section 2 and apply it to the dominating set problem in Section 3. As a result, we discover three boundary classes for the problem in question.

All graph classes in this paper are *hereditary*, i.e. closed under deletion of vertices. A hereditary class is called *monotone* if it is closed under deletion of edges. If a graph  $G$  does not contain any induced subgraph isomorphic to a graph in a set  $\mathbf{Y}$ , we say that  $G$  is  $\mathbf{Y}$ -free. The set of all  $\mathbf{Y}$ -free graphs will be denoted  $\text{Free}(\mathbf{Y})$ . It is well known that for every hereditary class of graphs  $\mathbf{X}$ , there is a set  $\mathbf{Y}$  such that  $\mathbf{X} = \text{Free}(\mathbf{Y})$ . The minimal set  $\mathbf{Y}$  with this property is unique and will be denoted  $\text{Forb}(\mathbf{X})$ . If  $\text{Forb}(\mathbf{X})$  is finite, we call  $\mathbf{X}$  a *finitely defined* class.

Given a graph  $G = (V, E)$  and a subset of vertices  $U \subseteq V$ , we denote by  $G - U$  the subgraph of  $G$  induced by  $V - U$ . The set of vertices adjacent to a vertex  $v \in V$  is denoted  $N(v)$  and is called the neighborhood of  $v$ . The degree of  $v$  is  $|N(v)|$ . As usual,  $C_n$  is the chordless cycle and  $K_n$  is the complete graph on  $n$  vertices. Also,  $2K_2$  is the disjoint union of two copies of  $K_2$ , and  $K_n - e$  is the graph obtained from  $K_n$  by deleting a single edge.  $K_{n,m}$  stands for the complete bipartite graph with parts of the size  $n$  and  $m$ . By  $H_i$  we denote the graph in Fig. 1(a).

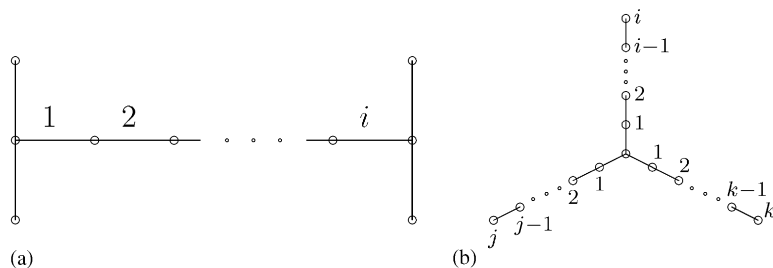
Throughout the paper we use special notations for two particular classes of graphs:

$\mathbf{Z}_k$  is the class of  $(C_3, \dots, C_k, H_1, \dots, H_k)$ -free graphs with maximum degree three,

$\mathbf{T}$  is the class of graphs every connected component of which is of the form  $T_{i,j,k}$  with some values of  $i, j, k \geq 0$  (Fig. 1(b)). Notice that if at least one of the indices  $i, j, k$  equals 0, then  $T_{i,j,k}$  is a path.

<sup>1</sup> Partially supported by the Russian Foundation for Basic Research (Grant 00-01-00601).

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Fig. 1. Graphs (a)  $H_i$  and (b)  $T_{i,j,k}$ .

In a graph  $G = (V, E)$ , a subset of vertices  $I \subseteq V$  is called *independent* if no two vertices in  $I$  are linked by an edge. A *clique* is a subset of pairwise adjacent vertices. A graph is called *split* if its vertices can be partitioned into an independent set and a clique. It has been proven in [5] that the class of split graphs is exactly  $\text{Free}(2K_2, C_4, C_5)$ .

## 2. Fundamentals of boundary classes

Let  $\Pi$  be a graph problem which is NP-hard in general graphs. A hereditary class of graphs  $\mathbf{X}$  will be called  $\Pi$ -hard if the problem  $\Pi$  remains NP-hard when restricted to graphs in  $\mathbf{X}$ . In case that  $\Pi$  has a polynomial time solution for graphs in  $\mathbf{X}$ , we shall say that  $\mathbf{X}$  is  $\Pi$ -easy.

**Definition 1.** A hereditary class of graphs  $\mathbf{X}$  will be called a limit class for  $\Pi$  if there exists a sequence  $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \dots$  of  $\Pi$ -hard classes such that  $\bigcap_{n \geq 1} \mathbf{X}_n = \mathbf{X}$ .

Notice that every  $\Pi$ -hard class is limit for  $\Pi$ . The following example shows that the converse is not true in general.

**Example 1.** Let  $\Pi$  be the independent set problem, i.e. the problem of finding in a graph an independent set of maximum cardinality, and  $\mathbf{X}_k := \text{Free}(C_3, \dots, C_k)$ . It has been proven in [8] that the independent set problem is NP-hard in the class  $\mathbf{X}_k$  for every fixed  $k \geq 3$ . It is not hard to see that  $\mathbf{X}_k \supseteq \mathbf{X}_{k+1}$  for each  $k$ , and  $\bigcap_n \mathbf{X}_n$  is the class of forests, i.e. graphs every connected component of which is a tree. Thus, the class of forests is a limit class for the independent set problem.

This example in conjunction with the well-known fact that the independent set problem has a polynomial time solution in the class of forests imply that a limit class for a problem  $\Pi$  is not necessarily  $\Pi$ -hard. However, if we restrict ourselves to finitely defined classes, then the two notions— $\Pi$ -hard and limit for  $\Pi$ —become equivalent.

**Lemma 1.** A finitely defined class  $\mathbf{X}$  is a limit class for  $\Pi$  if and only if it is  $\Pi$ -hard.

**Proof.** Let  $\text{Forb}(\mathbf{X}) = \{G_1, \dots, G_k\}$ , and  $\mathbf{X}$  be a limit class, i.e.  $\mathbf{X} = \bigcap_n \mathbf{X}_n$  for a sequence  $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \dots$  of  $\Pi$ -hard classes. Clearly, there must exist a natural  $n$  such that  $\mathbf{X}_n$  does not contain  $G_1, \dots, G_k$ . But then  $\mathbf{X}_i = \mathbf{X}$  for each  $i \geq n$ , and hence  $\mathbf{X}$  is  $\Pi$ -hard.  $\square$

The following two lemmas establish some important properties of limit classes.

**Lemma 2.** If  $\mathbf{X}$  is a limit class and  $\mathbf{Y} \supseteq \mathbf{X}$ , then  $\mathbf{Y}$  also is a limit class.

**Proof.** Let  $\mathbf{X} = \bigcap_n \mathbf{X}_n$ , where  $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \dots$  is a sequence of  $\Pi$ -hard graph classes. Then the class  $\mathbf{Y}_n := \mathbf{X}_n \cup \mathbf{Y}$  is  $\Pi$ -hard for every  $n$ ,  $\mathbf{Y}_1 \supseteq \mathbf{Y}_2 \supseteq \dots$ , and  $\mathbf{Y} = \bigcap_n \mathbf{Y}_n$ .  $\square$

**Lemma 3.** If  $\mathbf{X} = \bigcap_n \mathbf{X}_n$ , where  $\mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \dots$ , and  $\mathbf{X}_n$  is a limit class for each  $n$ , then  $\mathbf{X}$  is a limit class.

**Proof.** Let  $\text{Forb}(\mathbf{X}) = \{G_1, G_2, \dots\}$ . For every  $k$ , define  $\mathbf{X}^{(k)}$  to be the class  $\text{Free}(G_1, \dots, G_k)$ . Obviously, for every  $k$ , there exists an  $n$  such that  $\mathbf{X}_n$  does not contain  $G_1, \dots, G_k$ , and hence  $\mathbf{X}_n \subseteq \mathbf{X}^{(k)}$ . By Lemma 2,  $\mathbf{X}^{(k)}$  is a limit class, and by Lemma 1, it is  $\Pi$ -hard for each  $k$ . Obviously  $\mathbf{X}^{(k)} \supseteq \mathbf{X}^{(k+1)}$  and  $\bigcap_k \mathbf{X}^{(k)} = \mathbf{X}$ . Therefore,  $\mathbf{X}$  is a limit class.  $\square$

**Example 1** (continued). An important observation concerning the class of forests is that it is not a minimal limit class for the independent set problem. In fact, the paper [8] shows that the problem is NP-hard even for graphs with maximum degree three in the class  $\text{Free}(C_3, \dots, C_k)$ . Hence, the forests with vertex degree at most three constitute a limit class for the independent set problem. A stronger result has been obtained in [1]:

**Lemma 4.** *For every natural  $k \geq 3$ , the independent set problem is NP-hard in the class  $\mathbf{Z}_k$ .*

It is not hard to verify that  $\bigcap_k \mathbf{Z}_k$  coincides with the class  $\mathbf{T}$  defined in the introduction. Therefore,  $\mathbf{T}$  is a limit class for the independent set problem. Moreover, it has been recently proven in [2] that  $\mathbf{T}$  is a minimal under inclusion limit class for this problem (assuming that  $P \neq NP$ ). Minimal limit classes are of particular interest in our study. Below we introduce a special name for such classes and prove two results that explain our interest in them.

**Definition 2.** A minimal limit class for a problem  $\Pi$  will be called a boundary class for  $\Pi$ .

**Theorem 5.** *For every  $\Pi$ -hard class  $\mathbf{X}$ , there is a boundary class  $\mathbf{Y}$  for  $\Pi$  such that  $\mathbf{Y} \subseteq \mathbf{X}$ .*

**Proof.** Consider a bijection between the class of all graphs and the natural numbers. This bijection defines a linear order, which will be called *standard*. Let us define a sequence of graph classes  $\mathbf{X} = \mathbf{X}_1 \supseteq \mathbf{X}_2 \supseteq \dots$  as follows. Suppose that the class  $\mathbf{X}_n$  has been defined. Then we find the first graph  $G \in \mathbf{X}_n$  with respect to the standard order such that  $\mathbf{X}_n \cap \text{Free}(G)$  is a limit class. If there is no such  $G$ , then set  $\mathbf{X}_{n+1} := \mathbf{X}_n$ , otherwise set  $\mathbf{X}_{n+1} := \mathbf{X}_n \cap \text{Free}(G)$ .

Now consider the class  $\mathbf{Y} := \bigcap_n \mathbf{X}_n$ . Clearly  $\mathbf{Y} \subseteq \mathbf{X}$ . By Lemma 3,  $\mathbf{Y}$  is a limit class. In order to prove the minimality of  $\mathbf{Y}$ , assume there is a limit class  $\mathbf{Z}$  such that  $\mathbf{Z} \subset \mathbf{Y}$ . Let  $H \in \mathbf{Y} - \mathbf{Z}$ . Then  $\mathbf{Z} \subseteq \mathbf{Y} \cap \text{Free}(H) \subseteq \mathbf{X}_k \cap \text{Free}(H)$  for each  $k$ . Therefore, by Lemma 2,  $\mathbf{X}_k \cap \text{Free}(H)$  is a limit class for each  $k$ . For some  $k$ , the graph  $H$  becomes the first (under the standard order) graph with this property. But then  $\mathbf{X}_{k+1} := \mathbf{X}_k \cap \text{Free}(H)$ , and  $H$  belongs to no class  $\mathbf{X}_n$  with  $n > k$ , which contradicts  $H \in \mathbf{Y}$ .  $\square$

**Theorem 6.** *A finitely defined class is  $\Pi$ -hard if and only if it includes some boundary class for  $\Pi$ .*

**Proof.** If  $\mathbf{X}$  is a finitely defined class that includes a boundary class, then by Lemma 2,  $\mathbf{X}$  is a limit class, and by Lemma 1, it is  $\Pi$ -hard. The converse statement follows from Theorem 5.  $\square$

### 3. Boundary classes for the dominating set problem

In a graph  $G = (V, E)$ , a subset of vertices  $U \subseteq V$  is called *dominating* if every vertex of  $G$  outside  $U$  has a neighbor in  $U$ . The dominating set problem is that of finding in a graph, a dominating set of minimum cardinality. It is a well known NP-hard graph problem [6]. In the present section we describe three boundary classes for this problem. Our arguments are based on the following result obtained in [3].

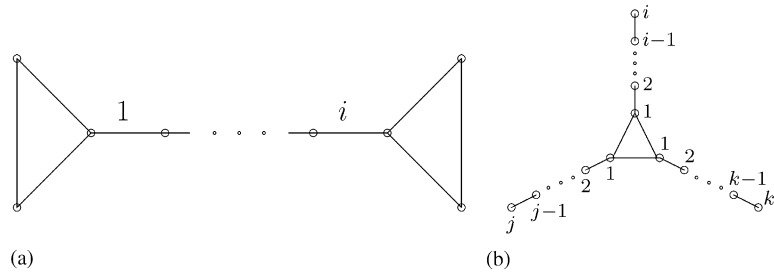
**Theorem 7.** *If  $\mathbf{X}$  is a monotone graph class and  $\mathbf{T} \not\subseteq \mathbf{X}$ , then the dominating set problem and the independent set problem can be solved for graphs in  $\mathbf{X}$  in polynomial time.*

**Theorem 8.** *If  $P \neq NP$ , then  $\mathbf{T}$  is a boundary class for the dominating set problem.*

**Proof.** It has been proven in [7] that the dominating set problem is NP-hard in the class  $\mathbf{Z}_k$  for any fixed  $k \geq 3$ . Since  $\mathbf{T} = \bigcap_{k \geq 3} \mathbf{Z}_k$ , we conclude that  $\mathbf{T}$  is a limit class for the problem. Now let us prove its minimality.

Assume by contradiction that a limit class  $\mathbf{X}$  is a proper subclass of  $\mathbf{T}$ , and let  $G$  be a graph in  $\mathbf{T} - \mathbf{X}$ . Then  $\mathbf{X} \subseteq \mathbf{T} \cap \text{Free}(G)$ . If  $G$  contains at least one connected component without vertices of degree three, we extend  $G$  to an arbitrary graph  $H \in \mathbf{T}$  in which every connected component has a vertex of degree three. Otherwise, set  $H := G$ . Since  $G$  is an induced subgraph of  $H$ , we have  $\text{Free}(G) \subseteq \text{Free}(H)$ . Now let  $\mathbf{M}$  denote the set of graphs containing  $H$  as a spanning subgraph. In other words, every graph in  $\mathbf{M}$ , other than  $H$ , is obtained from  $H$  by adding some edges. Obviously  $H$  is the only graph in  $\mathbf{T}$  belonging to  $\mathbf{M}$ , because addition of an edge to  $H$  results in appearing either a cycle or a graph  $H_i$  (Fig. 1(a)). We thus obtain the following inclusions:

$$\mathbf{X} \subseteq \mathbf{T} \cap \text{Free}(G) \subseteq \mathbf{T} \cap \text{Free}(H) \subseteq \text{Free}(\mathbf{M}).$$

Fig. 2. Graphs (a)  $\Phi_i$  and (b)  $T_{i,j,k}^A$ .

Clearly  $\mathbf{M}$  is a finite set, and therefore the dominating set problem is NP-hard in the class  $\text{Free}(\mathbf{M})$  by Lemmas 2 and 1. On the other hand,  $\text{Free}(\mathbf{M})$  is a monotone graph class and  $\mathbf{T} \not\subseteq \text{Free}(\mathbf{M})$ , since  $H \in \mathbf{T} - \text{Free}(\mathbf{M})$ . Consequently, the dominating set problem is polynomially solvable in the class  $\text{Free}(\mathbf{M})$  by Theorem 7. We have a contradiction with the assumption  $P \neq \text{NP}$ , which completes the proof.  $\square$

Denote by  $\mathbf{T}^A$  the class of graphs every connected component of which is of the form  $T_{i,j,k}^A$  with some values of  $i, j, k \geq 0$  (Fig. 2(b)). Similarly to  $T_{i,j,k}$ , the graph  $T_{i,j,k}^A$  may not have vertices of degree three if at least one of the indices  $i, j, k$  is 0.

**Theorem 9.** *If  $P \neq \text{NP}$ , then  $\mathbf{T}^A$  is a boundary class for the dominating set problem.*

**Proof.** Denote by  $\mathbf{Y}_k$  the class of  $(K_{1,3}, K_4 - e, K_4, C_4, \dots, C_k, \Phi_1, \dots, \Phi_k)$ -free graphs with vertex degree at most 3, where  $\Phi_i$  is the graph depicted in Fig. 2(a).

It has been proven in [7] that the dominating set problem is NP-hard in the class  $\mathbf{Y}_k$  for any particular value of  $k \geq 4$ . It is not hard to verify that  $\mathbf{T}^A = \bigcap_{k \geq 4} \mathbf{Y}_k$ . Hence  $\mathbf{T}^A$  is a limit class for the problem. In what follows we prove that  $\mathbf{T}^A$  is a minimal limit class.

Assume to the contrary that a limit class  $\mathbf{X}$  is properly contained in  $\mathbf{T}^A$ , and let  $G$  be a graph in  $\mathbf{T}^A - \mathbf{X}$ . Then  $\mathbf{X} \subseteq \mathbf{T}^A \cap \text{Free}(G)$ . Let  $H$  be a graph in  $\mathbf{T}^A$  such that  $G$  is an induced subgraph of  $H$ , and every connected component of  $H$  is of the form  $T_{j-1,j-1,j}^A$  with some value of  $j > 1$  (obviously any graph  $G \in \mathbf{T}^A$  can be extended to such graph  $H \in \mathbf{T}^A$ ). Since  $G$  is an induced subgraph of  $H$ , we have  $\text{Free}(G) \subseteq \text{Free}(H)$ . It is easy to see that one can delete an edge from each connected component of  $H$  in such a way that in the resulting graph, denoted  $H'$ , every connected component has the form  $T_{j,j,j}$ , i.e.  $H'$  belongs to  $\mathbf{T}$ . Now let  $\mathbf{M}$  be the set of all graphs containing  $H'$  as a spanning subgraph. It is not hard to verify that  $\mathbf{M} \cap \mathbf{T}^A = \{H\}$ . Hence, the following inclusions hold:

$$\mathbf{X} \subseteq \mathbf{T}^A \cap \text{Free}(G) \subseteq \mathbf{T}^A \cap \text{Free}(H) \subseteq \text{Free}(\mathbf{M}).$$

These inclusions in conjunction with Lemmas 2 and 1 imply NP-hardness of the dominating set problem in the class  $\text{Free}(\mathbf{M})$ , since  $\mathbf{M}$  is a finite set. On the other hand,  $\text{Free}(\mathbf{M})$  is a monotone graph class and  $\mathbf{T} \not\subseteq \text{Free}(\mathbf{M})$ , because  $H' \in \mathbf{T} - \text{Free}(\mathbf{M})$ . Therefore, the problem in question is polynomially solvable in the class  $\text{Free}(\mathbf{M})$  by Theorem 7. This contradiction completes the proof.  $\square$

To describe one more boundary class for the dominating set problem we introduce more notations and prove two auxiliary results.

For a graph  $G = (V, E)$ , denote by  $\mathcal{Q}(G)$  the split graph with the vertex set  $V \cup E$  and the edge set

$$\{(x, y) : x, y \in V, x \neq y\} \cup \{(x, e) : x \in V, e \in E \text{ and } e \text{ is incident to } x \text{ in } G\}.$$

With each hereditary class of graphs  $\mathbf{X}$  we associate a subclass of split graphs  $\mathbf{Q}(\mathbf{X})$  defined by  $\mathbf{Q}(\mathbf{X}) := \{\mathcal{Q}(G) : G \in \mathbf{X}\}$ . Notice that  $\mathbf{Q}(\mathbf{X})$  is not hereditary in general. By  $\mathbf{Q}^*(\mathbf{X})$  we denote the minimal hereditary class of graphs containing  $\mathbf{Q}(\mathbf{X})$ . In other words,  $\mathbf{Q}^*(\mathbf{X})$  is obtained by adding to  $\mathbf{Q}(\mathbf{X})$  all induced subgraphs of the graphs in  $\mathbf{Q}(\mathbf{X})$ . Obviously,  $\mathbf{Q}^*(\mathbf{X})$  is again a subclass of split graphs.

Let  $\mathbf{\Gamma}$  denote the class of all graphs. In the sequel, we shall use the following characterization of graphs in  $\mathbf{Q}^*(\mathbf{\Gamma})$ : a graph  $G$  belongs to  $\mathbf{Q}^*(\mathbf{\Gamma})$  if and only if the vertices of  $G$  can be partitioned into a clique  $C$  and an independent set  $I$

in such a way that

- (1) every vertex in  $I$  has degree at most 2;
- (2) any two different vertices of degree 2 in  $I$  have distinct neighborhoods.

The necessity of this characterization follows directly from the definition. The sufficiency can be seen from the following observation: every graph  $G$  that meets the characterization can be easily extended to a split graph  $H$  satisfying (1) and (2) with the additional requirement that every vertex in  $I$  has exactly two neighbors. Such graph  $H$  can be obtained in different ways. For instance, one can add to  $C$  a private neighbor for each vertex of degree one in  $I$ , and a couple of private neighbors for each isolated vertex in  $I$ . Clearly  $H \in \mathbf{Q}(\Gamma)$  and hence  $G \in \mathbf{Q}^*(\Gamma)$ .

From the above characterization we can easily derive the description of  $\mathbf{Q}^*(\Gamma)$  in terms of minimal forbidden induced subgraphs.

**Lemma 10.**  $\mathbf{Q}^*(\Gamma) = \text{Free}(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ , where  $K_{2,3} + e$  denotes the graph obtained from a  $K_{2,3}$  by linking its vertices of degree 3 by an edge.

**Proof.** Let  $G$  be a graph in  $\mathbf{Q}^*(\Gamma)$ . Then clearly  $G$  is  $(2K_2, C_4, C_5)$ -free, since it is a split graph [5]. Furthermore,  $G \in \text{Free}(K_5 - e, K_{2,3} + e)$  because neither  $K_5 - e$  nor  $K_{2,3} + e$  meet the above characterization. Thus,  $\mathbf{Q}^*(\Gamma) \subseteq \text{Free}(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ .

Conversely, let  $G \in \text{Free}(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ . From  $(2K_2, C_4, C_5)$ -freeness it follows that the vertices of  $G$  can be partitioned into a clique  $C$  and an independent set  $I$ . Among all such partitions let us consider any one in which  $C$  is a maximal under inclusion clique. Then condition (1) holds. Indeed, assume that a vertex  $v \in I$  has three neighbors  $a, b, c \in C$ . From the maximality of  $C$  we know that there is a vertex  $d \in C$  non-adjacent to  $v$ . But then  $a, b, c, d, v$  induce a  $K_5 - e$ . To prove (2), assume there are two vertices  $u$  and  $v$  in  $I$  such that  $N(u) = N(v) = \{a, b\} \subseteq C$ . By the maximality of  $C$  we conclude that there must be one more vertex in  $C$ , say  $c$ . But now  $a, b, c, u, v$  induce a  $K_{2,3} + e$ .  $\square$

**Lemma 11.** The independent set problem in a hereditary class of graphs  $\mathbf{X}$  is polynomially equivalent to the dominating set problem in the class  $\mathbf{Q}^*(\mathbf{X})$ .

**Proof.** The proof of the lemma is based on the following simple observation:

- (a) In a connected split graph  $G$  whose vertices can be partitioned into a clique  $C$  and an independent set  $I$ , there is a minimum dominating set which is a subset of  $C$ .

Indeed, if a minimum dominating set  $D$  in  $G$  contains a vertex  $u \in I$ , then any neighbor  $v \in C$  of  $u$  is not in  $D$  (else  $D - \{u\}$  is a smaller dominating set), and hence  $(D - \{u\}) \cup \{v\}$  is a minimum dominating set without  $u$ .

Now let  $G = (V, E)$  be a graph in  $\mathbf{X}$ , and  $Q(G)$  the respective split graph in  $\mathbf{Q}(\mathbf{X})$ . Notice that  $Q(G)$  is necessarily connected, and hence it contains a minimum dominating set  $D \subseteq V$ . It is not hard to verify that

- (b)  $D$  is dominating in  $Q(G)$  if and only if  $V - D$  is an independent set in  $G$ .

Hence the independent set problem in  $\mathbf{X}$  polynomially reduces to the dominating set problem in  $\mathbf{Q}(\mathbf{X})$ , and vice versa.

To complete the proof we have to show how the dominating set problem for graphs in  $\mathbf{Q}^*(\mathbf{X}) - \mathbf{Q}(\mathbf{X})$  can be reduced to the independent set problem for graphs in  $\mathbf{X}$ . Without loss of generality we may consider only connected graphs, because any isolated vertex belongs to every dominating set, and any split graph without isolated vertices is connected.

Let  $G$  be a connected graph in  $\mathbf{Q}^*(\mathbf{X}) - \mathbf{Q}(\mathbf{X})$  with a partition of its vertices into a clique  $C$  and an independent set  $I$  satisfying (1) and (2) (according to the arguments in the proof of Lemma 10 such partition can be found in polynomial time, because the number of maximal cliques in  $C_4$ -free graphs is bounded by a polynomial in the size of the graph [4], and hence the algorithm in [9] can generate all of them in polynomial time). Denote by  $J \subseteq I$  the set of vertices of degree 1, and by  $B \subseteq C$  the set of neighbors of the vertices in  $J$ . By (a), there is a minimum dominating set  $D$  in  $G$  such that  $D \subseteq C$ . Moreover,  $B \subseteq D$ , since otherwise some vertices in  $J$  are not dominated by  $D$ . Now let us create the graph  $G' = G - J \in \mathbf{Q}(\mathbf{X})$ , and the graph  $H = (C, I - J)$  such that  $Q(H) = G'$ . Finally, we find a maximum independent set  $I'$  in the graph  $H - B \in \mathbf{X}$ . Then, by (b),  $C - I'$  is a dominating set in  $G'$ . Moreover, it is a minimum dominating set containing  $B$ , since  $I' \cap B = \emptyset$ . Therefore,  $C - I'$  is a minimum dominating set of the graph  $G$ , which completes the reduction.

Such a reduction can also be derived from the result in [3] stating that the dominating set problem in a graph  $G$  polynomially reduces to the same problem for blocks in  $G$ . Obviously, any block in a connected graph  $G \in \mathbf{Q}^*(\mathbf{X})$  is either a  $K_2$  or a graph in  $\mathbf{Q}(\mathbf{X})$ .  $\square$

**Theorem 12.** *If  $P \neq NP$ , then  $\mathbf{Q}^*(\mathbf{T})$  is a boundary class for the dominating set problem.*

**Proof.** It is not hard to verify that  $\mathbf{Q}^*(\mathbf{T}) = \bigcap_k \mathbf{Q}^*(\mathbf{Z}_k)$ . Thus, from Lemmas 4 and 11 it follows that  $\mathbf{Q}^*(\mathbf{T})$  is a limit class for the dominating set problem. To prove its minimality, assume there is a limit class  $\mathbf{X}$  which is properly contained in  $\mathbf{Q}^*(\mathbf{T})$ . We consider a graph  $F \in \mathbf{Q}^*(\mathbf{T}) - \mathbf{X}$ , a graph  $G \in \mathbf{Q}(\mathbf{T})$  such that  $F$  is an induced subgraph of  $G$ , and a graph  $H \in \mathbf{T}$  such that  $G = Q(H)$ . From the choice of  $G$  and Lemma 10, we know that  $\mathbf{X} \subseteq \text{Free}(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ . Therefore, by Lemmas 2 and 1 the dominating set problem is NP-hard in the class  $\text{Free}(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ .

Denote by  $\mathbf{M}$  the set of all graphs containing  $H$  as a spanning subgraph. Clearly  $\text{Free}(\mathbf{M})$  is a monotone class, and  $\mathbf{T} \not\subseteq \text{Free}(\mathbf{M})$ . Consequently, the independent set problem has a polynomial time solution for graphs in  $\text{Free}(\mathbf{M})$  by Theorem 7, and hence the dominating set problem has a polynomial time solution for graphs in  $\mathbf{Q}^*(\text{Free}(\mathbf{M}))$  by Lemma 11.

To provide a contradiction let us show that

$$\mathbf{Q}^*(\text{Free}(\mathbf{M})) = \text{Free}(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e).$$

Consider first a graph  $H'$  in  $\mathbf{Q}^*(\text{Free}(\mathbf{M}))$  and let  $Q(H')$  denote a graph containing  $H'$  as an induce subgraph and belonging to  $\mathbf{Q}(\text{Free}(\mathbf{M}))$ , i.e.  $H' \in \text{Free}(2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ . Moreover,  $H'$  does not contain  $G$  as an induced subgraph, since otherwise  $H'$  contains  $H$  as a subgraph, which is forbidden. Thus,  $\mathbf{Q}^*(\text{Free}(\mathbf{M})) \subseteq \text{Free}(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$ .

To prove the converse inclusion, we observe that both  $\text{Free}(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e)$  and  $\mathbf{Q}^*(\text{Free}(\mathbf{M}))$  are subsets of  $\mathbf{Q}^*(\mathbf{T})$ . Therefore, it suffices to show that  $\text{Free}(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e) \cap \mathbf{Q}(\mathbf{T})$  is a subset of  $\mathbf{Q}(\text{Free}(\mathbf{M}))$ . Let  $Q(H')$  be a graph in  $\text{Free}(G, 2K_2, C_4, C_5, K_5 - e, K_{2,3} + e) \cap \mathbf{Q}(\mathbf{T})$ . Then  $H'$  does not contain  $H$  as a subgraph, for otherwise  $Q(H')$  would contain  $G$  as an induced subgraph. Therefore,  $H' \in \text{Free}(\mathbf{M})$ , and hence  $Q(H') \in \mathbf{Q}(\text{Free}(\mathbf{M}))$ .  $\square$

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